

Lectures on Contest Mathematics

US1 – Invitation to Contest Algebra and Geometry

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“Cogito ergo Sum” – “I think, therefore I am”

René Descartes (1596-1650)

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Chapter 1

Algebra

1.1 Operations rules with exponents

1. We write $4^3 = 4 \cdot 4 \cdot 4$ or $4^3 = 4 \times 4 \times 4$. The small raised number is called an *exponent*, and 4^3 is a *power of 4*. Write $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$ as a power of 4.
2. Write the product $4^3 \cdot 4^5$ as a power of 4.
3. [PEA Math1 Materials] Faced with the problem of multiplying 5^6 times 5^3 , Brook is having trouble deciding which of these four answers is correct: 5^{18} , 5^9 , 25^{18} , or 25^9 . Your help is needed. Once you have answered Brook's question, experiment with other examples of this type until you are able to formulate the *common-base principle for multiplication* of expressions $b^m \cdot b^n$.
4. [PEA Math1 Materials] Exponents are routinely encountered in scientific work, where they help investigators deal with large numbers:
 - (a) The human population of Earth is roughly 6000000000, which is usually expressed in *scientific notation* as 6×10^9 . The average number of hairs on a human head is 5×10^5 . Use scientific notation to estimate the total number of human head hairs on Earth.
 - (b) Light moves very fast — approximately 3×10^8 meters every second. At that rate, how many meters does light travel in one year, which is about 3×10^7 seconds long? This so-called *light year* is used in astronomy as a yardstick for measuring even greater distances.
5. [PEA Math1 Materials] In $7^4 \cdot 7^4 \cdot 7^4 = (7^4)^\Delta$ and $b^9 \cdot b^9 \cdot b^9 \cdot b^9 = (b^9)^\nabla$, replace the triangles by correct exponents. The expression $(p^5)^6$ means to write p^5 as a factor how many times? To rewrite this expression without exponents as $p \cdot p \cdot p \cdots$, how many factors would you need?
6. [PEA Math1 Materials] The diameter of an atom is so small that it would take about 10^8 of them, arranged in a line, to span one centimeter. It is thus a plausible estimate that a