Recursions!

A *recursion* consists of one or more starting values of a sequence of numbers along with a rule for generating future values of the sequence from ones you already know. For example, the sequence

$$1, 2, 4, 8, 16, \ldots$$

can be described by the recursion "The first term in the sequence is 1, and any term after the first in the sequence is twice the previous term."

More formally, if we label the terms in this sequence $a_0, a_1, a_2, a_3, \ldots$ then we can define the sequence by the recursion

$$\begin{aligned} a_0 &= 1 \\ a_{i+1} &= 2a_i. \end{aligned}$$

where the second equation holds for all $i \geq 1$. Notice that in this case we can also write an *explicit formula* for the Fibonacci numbers; we

Another famous sequence defined by a recursion is the *Fibonnacci sequence*, defined by

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_{n+2} &= F_{n+1} + F_n \end{aligned}$$

Exercise. Write out the first 10 terms of the Fibonacci sequence. Do you see any patterns or interesting properties of the numbers that occur in the sequence?

Another interesting recursion is at the heart of an interesting unsolved problem known as the Collatz conjecture. Consider a sequence of terms is formed by following the two rules below:

(a) If a term is even, divide by 2 to get the next term.

(b) If a term is odd, multiply by 3, then add 1, to get the next term.

Based on the definition, answer the following questions:

(1) Choose a positive integer of your like as the first term and compute the next few terms of the sequence. Does your sequence exhibit any interesting behavior?

(2) When we start with 12, the first few terms of the sequence are 12, 6, 3, 10, 5, 16, $\ldots$. What is the sum of the sequences first 2009 terms?

(3) Does the choice of seed value (first term) affect the answer to this question? Before you answer, you might want to consider the seed value 27?

(4) Let the first term be T_0, write T_{i+1} in terms of T_i for every positive integer i.

The Collatz Conjecture asks: given any starting value n, does the sequence T_i eventually contain the number 1?

Counting with recursions

Problem. In how many ways can a row of 10 squares be each colored either red or green in such a way that no two red squares are adjacent?

This problem, at first glance, does not appear to be about recursive sequences, and one can certainly sit down and write out all the possibilities. But we can generalize the problem to ask how many ways a row of n squares, rather than 10, can be colored either red or green in such a way that no two red squares are adjacent. For $n = 1$, there are two possibilities, R and G (here 'R' represents red and 'G' represents green). Let's look at it for $n = 2$. There are 3 possibilities in this case: RG, GR, and GG. What about for $n = 3$?

Let T_n be the number of ways to color the row when it is n squares long. Try to come up with a recursion for T_n. Then, use your recursion to calculate T_{10}!

Let's now consider one of the classic counting problems.

Problem. In how many ways can one choose a team of k people from the n people sitting in a room?

You are probably familiar with binomial coefficients; the answer to this problem is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. There is a nice recursion satisfied by these coefficients:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Can you find a combinatorial proof of this fact?

Exercise. Prove, using Pascal's recursion, that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

Problems

Here are some further problems for you to try.

1. Find recursions that describe the following sequences. Also find an explicit formula for the nth term in terms of n if you can.
   (a) $1, 1, 1, 1, 1, 1, 1, 1, \ldots$
   (b) $1, 4, 9, 16, 25, 36, 64, \ldots$
   (c) $1, 3, 5, 7, 9, 11, 13, \ldots$
   (d) $1, 2, 6, 24, 120, 720, 5040, \ldots$
   (e) $1, 1, 1, 3, 5, 9, 17, 31, \ldots$
2. Choose positive values for x_0 and x_1 that no one else will think of, then calculate seven more terms of the sequence defined recursively by $x_n = \frac{1+x_{n-1}}{x_{n-2}}$. What do you notice?
3. Choose a positive value for x_0 that no one else will think of, then calculate seven more terms of the sequence defined recursively by $x_n = \frac{1+x_{n-1}}{1-x_{n-1}}$. What do you notice?

4. (MathCounts 2006) A dresser has five drawers stacked vertically. To be able to reach the contents in an open drawer, no drawer that is adjacent to the open drawer may be open at the same time. In how many ways can one or more drawers be open so that the contents in each of the open drawers can be reached?

5. In how many ways can a 2×10 grid of squares be tiled with dominoes? (A domino covers two adjacent squares in the grid, and a tiling is a way of placing the dominoes so that every square is covered and no two dominoes overlap).

6. (AMC 10 2000) The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ starts with two 1s, and each term afterwards is the sum of its two predecessors. Which one of the ten digits is the last to appear in the units position of a number in the Fibonacci sequence?

7. (AMC 10B 2004) In the sequence $2001, 2002, 2003, \ldots$, each term after the third is found by subtracting the previous term from the sum of the two terms that precede that term. For example, the fourth term is $2001 + 2002 - 2003 = 2000$. What is the 2004^{th} term in this sequence?

8. (AMC 10B 2005) The first term of a sequence is 2005. Each succeeding term is the sum of the cubes of the digits of the previous term. What is the 2005^{th} term of the sequence?

9. (AMC 10B 2006) Let $a_1, a_2, \ldots$ be a sequence for which $a_1 = 2$, $a_2 = 3$, and $a_n = \frac{a_{n-1}}{a_{n-2}}$ for each positive integer $n \geq 3$. What is a_{2006}?

10. (AMC 12B 2009) Ten women sit in 10 seats in a line. All of the 10 get up and then reseat themselves using all 10 seats, each sitting in the seat she was in before or a seat next to the one she occupied before. In how many ways can women be reseated?

11. (HMMT 2009) Let F_n be the Fibonacci sequence, that is, $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$. Compute $\sum_{n=0}^{\infty} F_n/10^n$.

12. (AMC 12 2001) Consider sequences of positive real numbers of the form $x, 2000, y, \ldots$, in which every term after the first is 1 less than the product of its two immediate neighbors. For how many different values of x does the term 2001 appear somewhere in the sequence?

13. (AMC 12A 2009) The *tower function of twos* is defined recursively as follows: $T(1) = 2$ and $T(n+1) = 2^{T(n)}$ for $n \geq 1$. Let $A = (T(2009))^{T(2009)}$ and $B = (T(2009))^A$. What is the largest integer k such that $\underbrace{\log_2 \log_2 \log_2 \ldots \log_2}_{k \text{ times}} B$ is defined?

14. (AMC 12A 2008) Let $a_1, a_2, \ldots$ be a sequence determined by the rule $a_n = a_{n-1}/2$ if a_{n-1} is even and $a_n = 3a_{n-1} + 1$ if a_{n-1} is odd. For how many positive integers $a_1 \leq 2008$ is it true that a_1 is less than each of a_2, a_3, and a_4?

15. (AMC 12A 2008) A sequence $(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots$ of points in the coordinate plane satisfies
$$(a_{n+1}, b_{n+1}) = (\sqrt{3}a_n - b_n, \sqrt{3}b_n + a_n)$$
for $n = 1, 2, 3, \ldots$. Suppose that $(a_{100}, b_{100}) = (2, 4)$. What is $a_1 + b_1$?

16. (AIME 2008) The sequence (a_n) satisfies $a_1 = 1$ and

$$5^{(a_{n+1}-a_n)} - 1 = \frac{1}{n + \frac{2}{3}}$$

for $n \geq 1$. Let k be the least integer greater than 1 for which a_k is an integer. Find k.

17. (AIME 2001) Given that $x_1 = 211$, $x_2 = 375$, $x_3 = 420$, $x_4 = 523$, and $x_n = x_{n-1} - x_{n-2} + x_{n-3} - x_{n-4}$, when $x \geq 5$, find the value of $x_{531} + x_{753} + x_{975}$.

18. (AIME 2001) A set of positive numbers has the *triangle property* if it has three distinct elements that are the lengths of the sides of a triangle whose area is positive. Consider sets $\{4, 5, 6, \ldots, n\}$ of consecutive positive integers, all of whose ten-element subsets have the triangle property. What is the largest possible value of n?

19. (AIME 2003) Define a *good word* as a sequence of letters that consists only of the letters A, B, and C – some of these letters may not appear in the sequence – and in which A is never immediately followed by B, B is never immediately followed by C, and C is never immediately followed by A. How many seven-letter good words are there?

20. (AIME 2006) A collection of 8 cubes consists of one cube with edge-length k for each integer k, with $1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the rules:

    - Any cube may be the bottom cube of in the tower.
    - The cube immediately on top of a cube with edge-length k must have edge-length at most $k + 2$. Let T be the number of different towers that can be constructed. What is the remainder when T is divided by 1000?

21. (AMC 12A 2009) The first two terms of a sequence are $a_1 = 1$ and $a_2 = \frac{1}{\sqrt{3}}$. For $n \geq 1$,

$$a_{n+2} = \frac{a_n + a_{n+1}}{1 - a_n a_{n+1}}.$$

What is $|a_{2009}|$?

Other famous recursions

Here are some of my favorite recursions of all time:

- **The Catalan numbers:** The Catalan numbers C_n are defined by the recursion

$$\begin{aligned} C_0 &= 1 \\ C_n &= C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0 \end{aligned}$$

  The first few Catalan numbers are $1, 1, 2, 5, 14, 42, 132, \ldots$, and can also be described by the explicit formula $C_n = \frac{1}{n+1}\binom{2n}{n}$. This sequence comes up quite often in combinatorial problems (see, for instance, problem 13 below.)

- **The partition recursion:** Let $P(n,k)$ denote the number of ways of writing n as an (unordered) sum of k numbers, with repetitions allowed. Then the numbers $P(n,k)$, for $n \geq k$, satisfy the *two-variable recursion*

$$\begin{aligned} P(n,1) &= 1 \\ P(n,n) &= 1 \\ P(n+1,k+1) &= P(n-k,k+1) + P(n,k). \end{aligned}$$

- **The Ackermann Function:** The Ackermann function A requires several levels of recursion, and is a classic example of a very fast-growing function in a branch of logic known as computability theory. It is defined by the recursion

$$\begin{aligned} A(0,n) &= n+1 \\ A(m+1,0) &= A(m,1) \\ A(m+1,n+1) &= A(m, A(m+1,n)). \end{aligned}$$

Here are a few problems involving these recursions that you can try.

1. Prove that the nth Catalan number C_n counts the number of sequences of positive integers of length $2n+1$, starting and ending at 0, such that each number in the sequence is either one larger or one smaller than the previous. (For instance, $0,1,2,1,2,3,2,1,0$ is a valid sequence, but $0,1,2,1,0,-1,0$ is not.)
2. Make a table with the values of $P(n,k)$ for $n = 1,\ldots,10$ and $k = 1,\ldots,n$, and use the recursion for the partition function to calculate the values. Can you prove that this counts the number of parititions of n into k parts?
3. Make a table for the values of the Ackermann function $A(m,n)$ for the first few columns and as many rows as you need. At what point do the columns begin to grow very rapidly?