IDEA Math Admission Test Cover Sheet

Your name (please print) ____________________ ____________________

Last *First*

Contact Information __________________________ (phone number)

(please print) ____ (email address)

Number of pages (not including this cover sheet) ____________________

IDEA Math
May – August, 2008

Admissions Test

- The test has five sections. Each section has six problems, and it corresponds to a certain short course series. Consider courses you might want to take, and work on the problems in the corresponding sections. **If you can solve half or more of the chosen problems, we strongly encourage you to apply, but don't be discouraged if you can't.**
- You should include all significant steps in your reasoning and computation. We are interested in your ability to present your work, so unsupported answers will receive much less credit than well-reasoned progress towards a solution without a correct answer.
- In this document, you will find a cover sheet and an answer sheet. Print out each one, and make several copies of the blank answer sheet. Fill out the top of each answer sheet as you go, and then fill out the cover sheet when you're finished. **Start each problem on a new answer sheet.**
- **All the work you present must be your own**.
- **Don't be intimidated!** Some of the problems involve complex mathematical ideas, but all can be solved using only elementary techniques, admittedly combined in clever ways.
- **Be patient and persistent!** Learning comes more from struggling with problems than from solving them. Problem-solving becomes easier with experience. Success is not a function of cleverness alone.
- Make sure that the cover sheet is the first page of your submission, and that it is completely filled out.

  Solutions are to be mailed to the following address:

IDEA Math
P.O. Box 338
Exeter, NH 03833

If you e-mail your solutions, please send them to

application@ideamath.org

E-mailed solutions may be written and scanned or typed in LaTeX (or Tex). They should be sent as an attachment in either .doc or .pdf format. If you write and scan your solutions, insert the scans into a .doc or .pdf file, and send just the one file. If you have any questions, please feel free to contact us at **info@ideamath.org**

Please go the next page for the problems.

Admissions Test

§1. Math Roots Series

1.1. The sum of two positive integers is 60, and their least common multiple is 273. What is the product of the two numbers?

1.2. Digit d is randomly chosen from the set $\{4, 5, 6, 7\}$. After d is replaced, a digit e is randomly selected. What is the probability that the 2-digit number $\overline{de}$ is a multiple of 3?

1.3. In the addition shown below, different letters stand for different decimal digits.

$$\begin{array}{rcccc} & & T & H & E \\ +) & & K & E & Y \\ \hline & I & D & E & A \end{array}$$

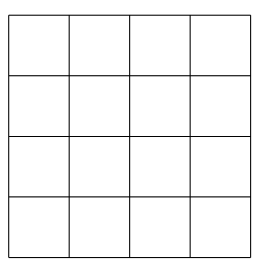
Find the maximum value of the 4-digit number $IDEA$.

1.4. Draw a regular pentagon and all five of its diagonals. How many triangles can you find in your picture? How many isosceles triangles can you find in your picture?

1.5. A *palindrome* is a number that has the property of reading the same in either direction, e.g., 1221 and 13431. Find the largest 5-digit palindrome that is divisible by 101.

1.6. A 4×4 chessboard can be tiled (no overlapping) by m pairwise incongruent rectangles with integer side lengths. Find the maximum value of m. Explain your reasoning.

(A square is a rectangle.)



§2. Beyond MATHCOUNTS Series

2.1. Let a, b, c, d be integers with $a < b < c < d$. Given that their product is 121, compute $a^c + b^d$.

2.2. Compute

$$2008(1 + 2 + \cdots + 2009) - 2009(1 + 2 + \cdots + 2008).$$

2.3. Let N be the largest integer for which both N and $7N$ have exactly 99 digits. What is the 50^{th} digit of N?

2.4. Given that $x^2 - 3x - 5 = 0$, compute $x^4 - 6x^3 + 9x^2 - 10$.

2.5. In triangle ABC, $AB = 4$, $BC = 5$, and $CA = 3$. Point D lies on side AB with $BD = 2$, and point E lies on side AC with $CE = 1$. Segments CD and BE meet at P. What is the positive difference between the areas of triangle BDP and CEP?

2.4. A collector has n precious stones. If the three heaviest stones are removed then the total weight of the stones decreases by 35%. From the remaining stones if the three lightest stones are removed the total weight further decreases by $\frac{5}{13}$. Determine, with explanation, the value of n.

§3. Math Circles Series

3.1. Sunny runs at a steady rate, and Moonbeam runs m times as fast, where m is a number greater than 1. If Moonbeam gives Sunny a head start of h meters, how many meters must Moonbeam run to overtake Sunny?

3.2. Let M, $1 \leq M \leq 100$, be an integer having the following property: if x is randomly chosen from the set of numbers $\{1, 2, \ldots, 100\}$, then the probability that x is a divisor of M is $\frac{1}{20}$. Find the maximum value of M.

3.3. What is the perimeter of a regular dodecagon whose area is $24 + 12\sqrt{3}$? (Hint: Show first that a regular dodecagon can be cut into pieces that are all regular polygons, which need not all have the same number of sides.)

3.4. A bowl contains a mixture of 2 red, 3 green, and 4 brown candies. Pat reaches into a bowl and grabs a handful of 6. What is the probability that a random 6-piece handful will have exactly 2 brown ones?

3.5. The vertices of a triangle are $A = (-3, 1)$, $B = (-2, -3)$, and $C = (x, 0)$. If the area of the triangle is 7.5, find x.

3.6. Five distinct numbers are given. When each pair of numbers are added together, a *complete* list of *different* sums that result is:

$$391,\ 423,\ 548,\ 669,\ 673,\ 705,\ 794, 826,\ 951.$$

Find number in the list is the sum of two different pairs of numbers?

§4. Advanced Topics Series

4.1. Let m and n be positive integers with $m \leq n$. An $m \times n$ rectangle is tiled by mn unit squares. A point is called *critical* if it is either an vertex of a unit square or the center of a unit square. Find (m, n) with minimal $m + n$ such that the rectangle has 865 critical points.

4.2. Numbers in the set $S = \{1, 2, \ldots, 15\}$ are divided into two nonempty sets S_1 and S_2 such that the product p_1 of the elements in set S_1 is divisible by the product p_2 of the elements in set S_2. Let m denote the minimum value of p_1/p_2. Find the remainder when m is divided by 1000.

4.3. Let n be the number of incongruent triangles such that its side lengths are all integers and the sum of the lengths of two of its sides is equal to 39. Find n.

4.4. Circle ω is centered at O and has radius 5. Points A and B lie on ω with $AB = 6$. Square $PQRS$ is inscribed in sector OAB with P on segment OA, Q on segment OB, R and S on minor arc $\widehat{AB}$. Find the area of $PQRS$.

4.5. If $x^2 + y^2 - 30x - 40y + 576 = 0$, then the largest possible value of $\frac{y}{x}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Determine $m + n$.

4.6. A particle moves by vectors on the coordinate-plane according to the following rule: (1) It can only be moved by vectors with nonnegative integer components; (2) If it is moved by vector $[m, n]$ to get to the current position, it will be moved by either $[m-1, n]$ or $[m, n-1]$. It starts at the origin $(0, 0)$ and is moved by $[13, 11]$ in its first move. When it can no longer move according to proceeding rules, it is located at point (r, r). Find r.

§5. Mathematical Proofs Series

5.1. Determine with proof all primes p such that $p^2 + 11$ has exactly six distinct positive integer divisors (including 1 and itself).

5.2. In trapezoid $ABCD$, $AB \parallel CD$ and $\angle C = 90°$. Let M be the midpoint of side AD. Suppose that $BM \perp AD$. If $BM = 12.5$ and $AB + BC + CD = 31$, find BC.

5.3. A rectangle $ABCD$ is divided into nine smaller rectangles by two lines parallel to side AB and two lines parallel to side AD. The areas of the four smaller rectangles at the corners order are 4, 12, 36, 108, and the area of the smaller rectangle at the center is 40. Determine smallest possible area of rectangle $ABCD$.

5.4. How many ways are there to write the digits $0, 1, 2, \ldots, 9$ in a row, such that each digit other than the left-most is within one of some digit to the left of it?

5.5. Consider the following two-person game. A number of pebbles are lying on a table. Two players make their moves alternately. A move consists in taking off the table x pebbles, where x is the square of any positive integer. The player who is unable to make a move loses. Prove that there are infinitely many initial situations in which the player who goes second has a winning strategy?

5.6. Prove that any n points in the plane can be covered by finitely many disks with the sum of the diameters less than n and the distance between any two disks greater than 1.

Suggestions for Writing Proofs
(By Tiankai Liu)

- All proofs should be written neatly and coherently in paragraphs of standard American English. Mathematical symbols like $\equiv$ and $\leq$ should be used only in equations, not as verbs or prepositions in a sentence. Do not write things like "all of the $\triangle$'s angles are $\leq 90°$"—this should be "$\angle A, \angle B, \angle C \leq 90°$" or "all of the triangle's angles are at most 90°." Avoid the symbols $\wedge$ $\vee$ $\because$ $\therefore$ $\forall$ $\exists$; instead, write out "and," "or," "because," "therefore," "for all," "there exists." Similarly, do not use $\Longrightarrow$ or $\Rightarrow\!\!\!\!\times\!\!\Leftarrow$ except as part of a sequence of equations.

- Write a statement using words rather than symbols unless this would be unnecessarily awkward. Do not invent more notation than is necessary to explain your solution.

- Write true statements. Do not write something that is only partially true, and then say how to fix it later. If you assume something in one of your statements, say clearly what you are assuming. Define all terms you make up. If you use figures, graphs, tables, etc., explain thoroughly what they represent.

- Use the following formats for common proof patterns:

  - *Proof by induction:* (1) state the claim, (2) check the base case, (3) prove the induction step, and (4) conclude with the words "induction is complete."
  - *Proof by contradiction:* (1) state the claim, (2) state that you are assuming the opposite, (3) derive a contradiction (and say why it is a contradiction), and (4) conclude that the claim follows.
  - *Proof by case analysis:* (1) state the claim, (2) state the various cases, (3) say why they exhaust all possibilities, (4) analyze the cases one by one, and (5) conclude that the claim follows.

- Draw accurate diagrams with compass and straightedge for geometry problems. This is for your own good as well as the graders'.

IDEA Math Admission Test Answer Sheet

Your name (please print) ____

Admissions Test   Problem Number ______   Page ___ of ___

Write neatly! Write all work inside the box. Do NOT write on the back of the page.