

IDEA Math Placement Test Cover Sheet

Your name (please print) _____

Last

First

IDEAMATH student ID number _____

Placement Test 1/2/3/4 (circle one)

Contact Information _____ (phone number)

_____ (email address)

Number of pages (not including this cover sheet) _____

Part	Recommended time	Actual time spent
Algebra	30 minutes	
Geometry	30 minutes	
Combinatorics and Number Sense	30 minutes	
Problem solving	60 minutes	

Placement Test

- The test has four sections. Each section has twenty-five problems, and it corresponds to a pair of course series. Consider courses you might want to take, and work on the problems in the corresponding sections. **If you can solve half or more of the chosen problems, we strongly encourage you to apply, but don't be discouraged if you can't.**
- You should include all significant steps in your reasoning and computation. We are interested in your ability to present your work, so unsupported answers will receive much less credit than well-reasoned progress towards a solution without a correct answer. **Do not simply submit a list of answers;** this makes it very difficult for us to evaluate your thinking and methods of approaching the problem.
- In this document, you will find a cover sheet and an answer sheet. Print out each one, and make several copies of the blank answer sheet. Fill out the top of each answer sheet as you go, and then fill out the cover sheet when you're finished. **Start each subsection on a new answer sheet.**
- **All the work you present must be your own.**
- **Don't be intimidated!** Some of the problems involve complex mathematical ideas, but all can be solved using only elementary techniques, admittedly combined in clever ways.
- **Be patient and persistent!** Learning comes more from struggling with problems than from solving them. Problem-solving becomes easier with experience. Success is not a function of cleverness alone.
- Make sure that the cover sheet is the first page of your submission, and that it is completely filled out.

Solutions are to be mailed to the following address:

IDEA Math
P.O. Box 338
Exeter, NH 03833

If you e-mail your solutions, please send them to

application@ideamath.org

E-mailed solutions may be written and scanned or typed in TeX. They should be sent as an attachment in either .doc or .pdf format. If you write and scan your solutions, insert the scans into a .doc or .pdf file, and send just the one file. If you have any questions, please feel free to contact us at info@ideamath.org.

Please go the next page for the problems.

Placement Test 4: ES1 and ES2

Algebra

A4.1 Evaluate: $\sum_{x=0}^{\infty} \frac{4x}{x^4 + 4}$

A4.2 Determine the number of ordered pairs (a, b) of integers such that $\log_a b + 6 \log_b a = 5$, $2 \leq a \leq 2005$, and $2 \leq b \leq 2005$.

A4.3 Let a, b, c, d be the roots of the polynomial $x^4 - 7x^3 + 3x^2 - 2$. Find the value of $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}$.

A4.4 Find and **prove** a closed formula for $1^3 + 2^3 + 3^3 + \dots + n^3$.

A4.5 How many positive integers n less than or equal to 1000 satisfy $(\sin t + i \cos t)^n = \sin nt + i \cos nt$ for all real t ?

Geometry

G4.1 The midpoints of two sides of a triangle are $(3, -1)$ and $(4, 3)$. One of the vertices of the triangle is $(7, -3)$. How many such triangles are there? Find the vertices of each of these triangles.

G4.2 Two circles of radius 6 intersect such that they share a common chord of length 6. The total area covered may be expressed as $a\pi + \sqrt{b}$, where a and b are integers. What is $a + b$?

G4.3 Let A, B and C be three distinct points on the graph of $y = x^2$ such that line AB is parallel to the x -axis and triangle ABC is a right triangle with area 2008. What is the y -coordinate of C ?

G4.4 Let ABC be a triangle and D a point on BC with $BD/DC = 5/7$. Let E be a point on AD with $AE/ED = 2/3$. Line CE meets segment AB at F . Find AF/FB .

G4.5 Triangle ABC has side lengths $AB = 65$, $BC = 33$, and $AC = 56$. Find the radius of the circle tangent to sides AC and BC and to the circumcircle of triangle ABC .

Combinatorics and Number Sense

C4.1 A standard six-sided die is rolled until three aces have appeared. Which is more likely, that the third ace appears on the twelfth roll, or that the third ace appears on the thirteenth roll? Carry out your computations in exact values. You can use your calculator to confirm your answer, but calculator data gets no credit in your solution.

C4.2 How many of the integers between 1 and 2011, inclusive, can be written as the difference of two perfect squares? Justify your answer.

C4.3 Determine the number of ways to arrange 6 red balls, 3 green balls, and 3 blue balls in a row with no adjacent balls of the same color. (Assume that balls of the same color are indistinguishable.)

C4.4 How many pairs of positive integers (a, b) are there such that $\gcd(a, b) = 1$ and

$$\frac{a}{b} + \frac{14b}{9a}$$

is an integer?

C4.5 For each positive integer n , let $f(n) = n^4 - 360n^2 + 400$. What is the sum of all values of $f(n)$ that are primes?

Problem Solving

P4.1 Find an explicit formula for each term a_n in the series of positive integers $a_1, a_2, a_3, a_4, \dots$, where $a_1 = 1, a_2 = 2$ and

$$2a_n a_{n-2} - a_{n-1}^2 = 2a_{n-1} a_{n-2}$$

for all positive integers $n \geq 3$.

P4.2 How many different 4×4 arrays, whose entries are all 1's and -1 's, have the property that the sum of the entries in each row is 0 and the sum of the entries in each column is 0?

P4.3 For how many pairs of integers (n, k) does $\binom{n}{k}$ have exactly three decimal digits?

P4.4 Let $ABCD$ be a rectangle with $AB = 10$ and $BC = 12$. Let M be the midpoint of CD , and P be the point on BM such that $DP = DA$. Find the area of quadrilateral $ABPD$.

P4.5 A *partition* of an integer n into k parts is a set of positive integers $\{a_1, a_2, a_3, \dots, a_k\}$ with $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_k$ and $a_1 + a_2 + a_3 + \dots + a_k = n$. Prove that the following three numbers are the same:

- (a) The number of partitions of n into at most k parts.
- (b) The number of partitions of $n + k$ into exactly k parts.
- (c) The number of partitions of n into parts, where each part has size at most k .

P4.6 Let S be the set of all points (x, y) in the coordinate plane such that $0 < x < \frac{\pi}{2}$ and $0 < y < \frac{\pi}{2}$. What is the area of the subset of S for which

$$\sin^2 x - \sin x \sin y + \sin^2 y \leq \frac{3}{4}?$$

P4.7 Four spheres, each of radius r , lie inside a regular tetrahedron with side length 1 such that each sphere is tangent to three faces of the tetrahedron and to the other three spheres. Find r .

P4.8 If a, b, c are distinct real numbers, prove that there cannot be a real number x such that the three equations $ax^2 + 2bx + c = 0$, $bx^2 + 2cx + a = 0$, and $cx^2 + 2ax + b = 0$ are all true.

P4.9 Evaluate: $\sum_{(i,j,k), i+j+k=19} ijk$ for non-negative integers i, j, k .

P4.10 Let $P_1 P_2 \dots P_{18}$ be a regular 18-gon. Prove that $P_1 P_{11}, P_5 P_{13}, P_8 P_{15}, P_9 P_{16}$ are concurrent.

Suggestions for Writing Proofs

(By Tiankai Liu)

- All proofs should be written neatly and coherently in paragraphs of standard American English. Mathematical symbols like \equiv and \leq should be used only in equations, not as verbs or prepositions in a sentence. Do not write things like “all of the \triangle 's angles are $\leq 90^\circ$ ”—this should be “ $\angle A, \angle B, \angle C \leq 90^\circ$ ” or “all of the triangle's angles are at most 90° .” Avoid the symbols $\wedge \vee \therefore \forall \exists$; instead, write out “and,” “or,” “because,” “therefore,” “for all,” “there exists.” Similarly, do not use \implies or $\implies\Leftarrow$ except as part of a sequence of equations.
- Write a statement using words rather than symbols unless this would be unnecessarily awkward. Do not invent more notation than is necessary to explain your solution.
- Write true statements. Do not write something that is only partially true, and then say how to fix it later. If you assume something in one of your statements, say clearly what you are assuming. Define all terms you make up. If you use figures, graphs, tables, etc., explain thoroughly what they represent.
- Use the following formats for common proof patterns:
 - *Proof by induction*: (1) state the claim, (2) check the base case, (3) prove the induction step, and (4) conclude with the words “induction is complete.”
 - *Proof by contradiction*: (1) state the claim, (2) state that you are assuming the opposite, (3) derive a contradiction (and say why it is a contradiction), and (4) conclude that the claim follows.
 - *Proof by case analysis*: (1) state the claim, (2) state the various cases, (3) say why they exhaust all possibilities, (4) analyze the cases one by one, and (5) conclude that the claim follows.
- Draw accurate diagrams with compass and straightedge for geometry problems. This is for your own good as well as the graders'.

IDEA Math Placement Test Answer Sheet

Your name (please print) _____

Placement Test Last 1/2/3/4 (circle one) First Page ___ of ___

Write neatly! Write all work inside the box. Do NOT write on the back of the page.