

Complex Numbers

Many quadratic equations have no real solutions. The simplest example is $x^2 + 1 = 0$. The modern approach is to let i stand for a number that has the property $i^2 = -1$. The solutions to $x^2 + 1 = 0$ are therefore $x = i$ and $x = -i$, and the solutions to $x^2 - 4x + 5 = 0$ are $x = 2 + i$ and $x = 2 - i$.

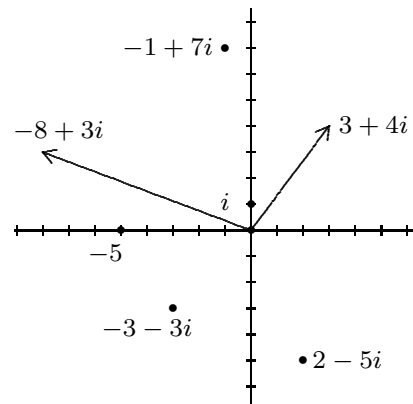
1. By completing the square, find the solutions to $x^2 - 6x + 13 = 0$, expressing them in the same $a + bi$ form.

Assuming that a and b are ordinary real numbers, an expression of the form $a + bi$ is called a *complex number*. You will sometimes hear $a + bi$ called an *imaginary number*, but this is somewhat misleading, for an ordinary real number also takes this form (for example, 3 is the same as $3 + 0i$). The number i is called the *imaginary unit*, and bi is called *pure imaginary* if b is a nonzero real number. The rules of algebra still hold for this system of numbers. These rules, together with the new fact $i^2 = -1$, allow you to manipulate algebraic expressions. For example, $(3 + 2i) - (1 + 7i)$ is equivalent to $2 - 5i$, and $(1 + i)(2 - i)$ can be expanded and simplified to $3 + i$.

2. Simplify each of the following to standard $a + bi$ form:

(a) $3 + 4i - 1 - i + 6$ (b) $(2 + i)(3 + i)$ (c) $(1 + i)^2$ (d) i^{95} (e) $(-1 + i\sqrt{3})^3$

3. Whatever these numbers may represent, it is important to be able to visualize them. Here is how to do it: The number $a + bi$ is matched with the point (a, b) , or with the vector $[a, b]$ that points from the origin to (a, b) . Points $(0, y)$ on the y -axis are thereby matched with pure imaginary numbers $0 + yi$, so the y -axis is sometimes called the *imaginary axis*. The x -axis is called the *real axis* because its points $(x, 0)$ are matched with real numbers $x + 0i$. The real-number line can thus be thought of as a subset of the *complex-number plane*. Plot the following complex numbers: $1 + i$, $-5i$, and $1 + i\sqrt{3}$.



4. Because complex numbers have two components, which are frequently referred to as x and y , it is traditional to use another letter to name complex numbers, as in $z = x + yi$. The components of $z = x + yi$ are usually called the *real part of z* and the *imaginary part of z* . Notice that i is not included in the imaginary part; for example, the imaginary part of $3 - 4i$ is -4 . Thus the imaginary part of a complex number is a real number.

- (a) What do we call a complex number whose imaginary part is 0?
- (b) What do we call a complex number whose real part is 0?
- (c) Describe the configuration of complex numbers whose real parts are all 2.
- (d) Describe the configuration of complex numbers having equal real and imaginary parts.

5. Because complex numbers are identified with vectors in the plane, it makes sense to talk about their *magnitudes*. Calculate (a) $|3 + 4i|$ (b) $|i|$ (c) $|3 - 2i|$ (d) $|3 + 2i|$

6. Describe the configuration of complex numbers whose magnitude is 5. Give examples.

7. It makes sense to regard $3 + 2i$ and $-4 + 6i$ as *perpendicular*. Explain.

Complex Numbers

As you have already seen, complex numbers belong both to algebra (which is how they were discovered) and to geometry (which was a much later development). The addition and subtraction of complex numbers treats them as though they were vectors. Multiplication seems mysterious at first, but it also has a geometric basis.

1. Multiply $5 + 2i$ by (a) 2; (b) i ; (c) $2i$; and describe the results geometrically.
2. Show that the triangle formed by the vectors $3 + i$ and $2 + 2i$ is *similar* to the triangle formed by the vectors $(1 + 2i)(3 + i)$ and $(1 + 2i)(2 + 2i)$. (Draw all four vectors with their tails at the origin.)

For a complete understanding of complex multiplication and division, it is necessary to deal with arbitrary angles. Therefore circular functions now make their appearance.

3. The useful abbreviation $\text{cis } \theta$ stands for the complex number $\cos \theta + i \sin \theta$. Where have you seen cosine and sine combined in this way before? Explain the abbreviation *cis*, then show that each of the following can be written in the form $\text{cis } \theta$:

(a) $0.6 + 0.8i$ (b) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (c) $0.28 + 0.96i$

Thus a complex number can be expressed in *rectangular form*, which means $a + bi$, or in *polar form*, which means $r \text{cis } \theta$.

4. Find the rectangular form that is equivalent to $2 \text{cis } 72$. Find the polar form that is equivalent to $3 + 4i$.
5. Write $3 + 4i$ and $1 + i$ in polar form. Then calculate the product of $3 + 4i$ and $1 + i$, and write this complex number in polar form as well. Do you notice anything remarkable?
6. The product of the complex numbers $\text{cis}(35)$ and $\text{cis}(21)$ can itself be written in the form $\text{cis } \theta$. What is θ ? What is the product of $4 \text{cis}(35)$ and $3 \text{cis}(21)$?
7. In degree mode, the numbers $\text{cis } 90$, $\text{cis } 180$, $\text{cis } 270$, and $\text{cis } 360$ have much simpler names. What are they?
8. Multiply $a + bi$ times $c + di$. Consider the special case when $a = \cos \theta$, $b = \sin \theta$, $c = \cos \phi$, and $d = \sin \phi$. You should see a couple of familiar formulas. Use them to prove the angle-addition property of complex multiplication.
9. Show that the product of the two non-real numbers $3 + 2i$ and $3 - 2i$ is a positive real number. Assuming that a and b are real numbers, explain why the product $(a + bi)(a - bi)$ is always a nonnegative real number.
10. Given a complex number $z = a + bi$, its *conjugate* is the number $\bar{z} = a - bi$.
 - (a) What geometric transformation is being applied?
 - (b) What happens when a complex number and its conjugate are multiplied?
 - (c) What happens when a complex number and its conjugate are added?
 - (d) Conjugation distributes over all arithmetic. Prove that $\overline{w \cdot z} = \bar{w} \cdot \bar{z}$, for example.

Complex Numbers

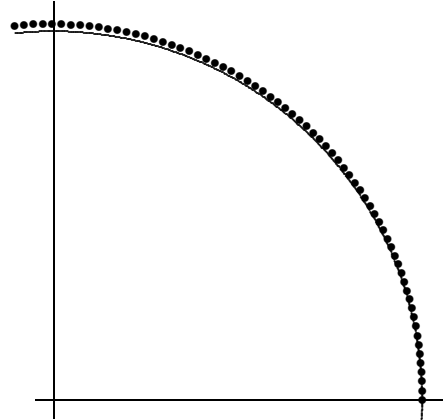
1. What is $i^{1234567890}$?
2. Show that the expression $\frac{1}{1+i}$ can be written in $a + bi$ form. In other words, show that the reciprocal of a complex number is also a complex number.
3. Show that the quotient $\frac{7+24i}{2+i}$ can be simplified to $a + bi$ form.
4. Write $(1+i)^8$ in standard $a + bi$ form.
5. Write $(2+i)^{-1}$, $(2+i)^0$, $(2+i)^1$, $(2+i)^2$, $(2+i)^3$, and $(2+i)^4$ in $a + bi$ form. Graph these numbers, and write them in polar form. What patterns do you notice?
6. A truly remarkable property of complex multiplication is the angle-addition identity $\text{cis}(\alpha)\text{cis}(\beta) = \text{cis}(\alpha + \beta)$. From this follows the *theorem of De Moivre*, which says that $(r \text{cis } \theta)^n = r^n \text{cis}(n\theta)$ for any numbers r and θ , and any integer n .
 - (a) Without using a calculator, find $(\sqrt{3} + i)^{18}$.
 - (b) Write the identity that covers complex division. What is the reciprocal of $\text{cis}(\alpha)$?
7. You may have noticed that the identity $\text{cis}(\alpha)\text{cis}(\beta) = \text{cis}(\alpha + \beta)$ looks exactly like another familiar rule you have learned about. What rule?

8. Show that $2 + i$ is one of the square roots of $3 + 4i$. What is the other square root?
9. Perhaps you know already that $\sqrt{3 + 4i}$ is an ambiguous expression. Explain why the same is true of the expressions $\sqrt{(0.1 + i)^2}$, $\sqrt{(-0.1 + i)^2}$, and $\sqrt{-1}$.
10. The quadratic equation $2z^2 + 2iz - 5 = 0$ has two solutions. Find them. By the way, the quadratic formula is still useful for non-real quadratic equations.
11. Express all the solutions to the following equations in $a + bi$ form:
 - (a) $z^2 + 2z + 4 = 0$
 - (b) $z^3 - 8 = 0$
 - (c) $2z + iz = 2 - i$
12. Verify that $z = 1 + i$ is one of the roots of $z^4 + 4 = 0$. Find the other three and then graph them all.

13. Working in degree mode, simplify the sum $\text{cis } 72 + \text{cis } 144 + \text{cis } 216 + \text{cis } 288 + \text{cis } 360$.
14. Consider the geometric sequence defined recursively by $z_0 = 1$ and $z_n = (4 + i)z_{n-1}$ for positive n . This sequence enters the second quadrant when $n = 7$, then leaves it. For what value of n does the sequence return to the second quadrant for the second time?
15. Find the sum of the infinite geometric series $\sum_{n=0}^{\infty} \left(\frac{1+i}{2}\right)^n$. Now think of the series as a sum of vectors (placed head-to-tail), and draw a spiral to confirm your answer.
16. Show that $\frac{7+4i}{8-i}$ is on the unit circle.

Complex Numbers and Recursion

1. The figure at right shows a few dozen terms of the infinite sequence $z_k = \left(1 + \frac{i}{40}\right)^k$. The points follow a spiral path that unwinds slowly from the unit circle (shown in part) as k increases. Notice that $z_0 = 1$ is on the unit circle and that the rest of the dots are outside. (a) Confirm this by calculating the magnitude of z_{40} . (b) Calculate the polar angle of z_{40} , thinking of z_{40} as a vector that emanates from the origin. (c) The dots have left the first quadrant. Find the next k for which z_k is again in the first quadrant.

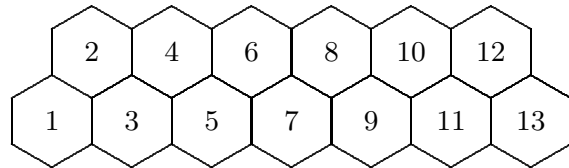


2. Given that $(a + bi)(3 - 4i) = 100$, where a and b are real numbers, what are the values of a and b ?
3. Asked on a test to simplify i^{83} , Cameron wrote the following solution:

$$i^{83} = (\sqrt{-1})^{83} = \left((-1)^{1/2}\right)^{83} = \left((-1)^{83}\right)^{1/2} = (-1)^{1/2} = \sqrt{-1} = i$$

What do you think of this solution?

4. Consider the honeycomb pattern shown at right, which consists of two rows of numbered hexagons. A honeybee crawls from hexagon number 1 to hexagon number 13, always moving from one hexagon to an adjacent hexagon whose number is greater. How many different paths are possible?



5. Find both square roots of $15 + 8i$, using any method you like.
6. Working in degree mode, evaluate
 (a) $\text{cis } 11 + \text{cis } 83 + \text{cis } 155 + \text{cis } 227 + \text{cis } 299$ and
 (b) $\cos 11 + \cos 83 + \cos 155 + \cos 227 + \cos 299$.
 Find a way to answer these questions without using your calculator.
7. Show that $(ac - bd)^2 + (ad + bc)^2$ can be written in factored form.
8. It is a fact that 149 and 130 can each be written as a sum of two square integers. Show that $19370 = 130 \cdot 149$ can also be written as a sum of two square integers, and that there is more than one way to do it.
9. Suppose that m and n are positive integers that can each be written as a sum of two square integers. Show that mn can also be written as a sum of two square integers.
10. Notice that $(2 + i)(3 + i) = 5 + 5i$. Working in degree mode, use this result to help you simplify $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$.
11. Simplify the expressions $(3 + 4i)(7 - i)$ and $(a + (a + 1)i)(2a + 1 - i)$.
12. Working in degree mode, simplify the expression $4 \tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$.

Complex Numbers and Recursion

1. A solution to $z^3 = 1$ is (of course) a cube root of 1. It is also called a *third root of unity*. Express these three numbers in $a + bi$ form and in polar form.
2. There are five fifth roots of unity. One of them is $z = 1$. The other four fifth roots come from a quartic polynomial with integer coefficients. Find that polynomial.
3. Explain why you would expect the polynomial

$$p(z) = z^{11} + z^{10} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1$$

to be divisible by the polynomial $q(z) = z^2 + z + 1$. Complete the factorization of $p(z)$ into polynomials with integer coefficients.

4. Suppose that z is both a 20^{th} root of unity and a 36^{th} root of unity. Explain why z must in fact be a fourth root of unity.

5. [AIME 1992] Consider the region A in the complex plane that consists of all points z such that both $z/40$ and $40/\bar{z}$ have real and imaginary parts between 0 and 1 inclusive. What is the integer that is nearest the area of A ?

6. [AIME 1984] Find the value of $10 \cot \left(\cot^{-1} \frac{1}{3} + \cot^{-1} \frac{1}{7} + \cot^{-1} \frac{1}{13} + \cot^{-1} \frac{1}{21} \right)$.

7. [AIME 1985] Given that $(a + bi)^3 - 107i = c$, where a , b , and c are integers, find c .

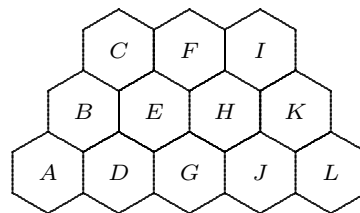
8. [AIME 1994] The points $O = (0, 0)$, $A = (a, 11)$, and $B = (b, 37)$ are the vertices of an equilateral triangle. Find the value of ab .

9. If all the coefficients of a polynomial equation $p(x) = 0$ are real, then non-real solutions of this equation must occur in conjugate pairs. Explain.

10. [AIME 1995] The equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four non-real roots, where a , b , c , and d are all real numbers. The product of two of the roots is $13 + i$ and the sum of the other two roots is $3 + 4i$, where $i^2 = -1$. Find b .

11. [AIME 1999] A function f is defined on the complex numbers by $f(z) = (a + bi)z$, where a and b are real numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a + bi| = 8$ and $b^2 = m/n$, where m and n are relatively prime positive integers, find $m + n$.

12. A honeybee crawls from hexagon A to hexagon L , always moving from one hexagon to an *adjacent* hexagon whose label comes *later* in the alphabet. For example, the only possible moves from cell E are to cells F , H , and G . How many different paths are there from A to L ?



13. [AIME 1999] Given that $\sum_{k=1}^{35} \sin 5k = \tan \frac{m}{n}$, where m and n are relatively prime positive integers and angles are measured in degrees, find $m + n$.

Complex Numbers and Recursion

- [AIME 2002] Let $F(z) = \frac{z+i}{z-i}$. If $z_0 = \frac{1}{137} + i$, and $z_n = F(z_{n-1})$ for all positive integers n , then $z_{2002} = a + bi$, where a and b are real numbers. Find $a + b$.
- Choose any two positive numbers a and b , and use them to generate an infinite sequence as follows: Let $x_1 = a$, $x_2 = b$, and $x_n = (1 + x_{n-1})/x_{n-2}$ for all subscripts $n > 2$. What is x_{2008} ?
- Choose any two numbers x_1 and x_2 , and generate an infinite sequence recursively by defining $x_n = |x_{n-1}| - x_{n-2}$ for all subscripts $n > 2$. What is x_{2008} ?
- Let $f_1(x) = \frac{x-1}{x}$, and $f_n(x) = f_1(f_{n-1}(x))$ for all subscripts $n > 1$. Find the value of $f_{2008}(2008)$.
- [AIME 1983] Given that $w^2 + z^2 = 7$ and $w^3 + z^3 = 10$, find the maximum value of $|w + z|$.
- [AIME 2007] The complex number z is equal to $9 + bi$, where b is a positive real number and $i^2 = -1$. Given that the imaginary parts of z^2 and z^3 are equal, find b .
- [AHSME 2004] A function f is defined by $f(z) = i\bar{z}$, where $i^2 = -1$ and \bar{z} is the conjugate of z . How many values of z satisfy both $|z| = 5$ and $f(z) = z$?
(A) 0 (B) 1 (C) 2 (D) 4 (E) 8
- [AHSME 1993] Consider the equation $10z^2 - 3iz - k = 0$, where $i^2 = -1$. Which of the following statements is true?
(A) For all positive real numbers k , both roots are pure imaginary.
(B) For all negative real numbers k , both roots are pure imaginary.
(C) For all pure imaginary numbers k , both roots are real and rational.
(D) For all positive real numbers k , both roots are real and irrational.
(E) For all complex numbers k , neither root is real.
- [AHSME 1992] Let $i^2 = -1$. Define a sequence of complex numbers by $z_1 = 0$ and $z_n = z_{n-1}^2 + i$ for all positive n . In the complex plane, how far from the origin is z_{111} ?
(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) $\sqrt{110}$ (E) $\sqrt{111}$
- [AHSME 1987] It is given that all the roots of $z^4 + az^3 + bz^2 + cz + d = 0$ are complex numbers that lie on the unit circle $|z| = 1$, where a, b, c , and d are real numbers. The sum of the reciprocals of the roots is necessarily
(A) a (B) b (C) c (D) $-a$ (E) $-b$
- [HMMT 2005] Let $A_1A_2A_3 \cdots A_k$ be a regular k -sided polygon inscribed in a circle of radius 1. Find the largest possible value of $PA_1 \cdot PA_2 \cdot PA_3 \cdots PA_k$, where P ranges over all points that are not outside the circle.
- [HMMT 2005] Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial whose coefficients are real, and whose roots are all roots of unity other than 1 itself. Show that the coefficients of p are symmetric. In other words (assuming that $a_n \neq 0$), show that $a_j = a_{n-j}$ holds for all $0 \leq j \leq n$.

Complex Numbers and Recursion

1. [R. Devaney] Consider the nonlinear two-dimensional recursion

$$\begin{aligned}x_{n+1} &= 1 + |x_n| - y_n \\ y_{n+1} &= x_n\end{aligned}$$

- (a) The sequence initiated by $(x_0, y_0) = (0.7, 1.7)$ is periodic. Calculate its period.
(b) The sequence initiated by $(x_0, y_0) = (5.0, 3.0)$ is periodic. Calculate its period.
(c) It can be shown (with some difficulty) that every sequence produced by this recursion is *bounded* (its points all land in a suitably large rectangle). Use this fact to explain why such a sequence is necessarily periodic if x_0 and y_0 are both rational.
2. One of the fifth roots of the complex number c is $2 \operatorname{cis} 18$. Find c , and then express its other fifth roots using polar (cis) notation.
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3. Let $P_0P_1P_2$ be an equilateral triangle inscribed in a circle of radius 1, and calculate the product $P_0P_1 \cdot P_0P_2$.

4. Let $P_0P_1P_2P_3$ be a square inscribed in a circle of radius 1, and calculate the product $P_0P_1 \cdot P_0P_2 \cdot P_0P_3$.

5. Let $P_0P_1P_2P_3P_4$ be a regular pentagon inscribed in a circle of radius 1, and calculate the product $P_0P_1 \cdot P_0P_2 \cdot P_0P_3 \cdot P_0P_4$.

6. Let $P_0P_1P_2P_3P_4P_5$ be a regular hexagon inscribed in a circle of radius 1, and calculate the product $P_0P_1 \cdot P_0P_2 \cdot P_0P_3 \cdot P_0P_4 \cdot P_0P_5$.

7. Let $P_0P_1P_2P_3P_4P_5P_6$ be a regular heptagon inscribed in a circle of radius 1, and calculate the product $P_0P_1 \cdot P_0P_2 \cdot P_0P_3 \cdot P_0P_4 \cdot P_0P_5 \cdot P_0P_6$.

- n. Let $P_0P_1P_2 \dots P_{n-1}$ be a regular n -gon inscribed in a circle of radius 1, and calculate the product $P_0P_1 \cdot P_0P_2 \cdots P_0P_{n-1}$.
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8. [HMMT 2005] Find a real, quartic polynomial that has integer coefficients, that is *irreducible* (which means that it cannot be factored further into polynomials with integer coefficients), and that has only 12^{th} roots of unity as its roots.

9. [HMMT 2005] Suppose that w and z are two k^{th} roots of unity. Prove that $(w + z)^k$ is a real number.

10. [HMMT 2005] Let \mathcal{P} be a regular k -sided polygon inscribed in a circle of radius 1. Find the sum of the squares of the lengths of all the sides and diagonals of \mathcal{P} .

11. [AHSME 1980] Given that $i^2 = -1$, what is the largest positive integer n for which $(n + i)^4$ is an integer?

Answers

page 1

1. $3 \pm 2i$ 2a. $8 + 3i$ 2b. $5 + 5i$ 2c. $2i$ 2d. $-i$ 2e. 8 3a. real 3b. pure imaginary, if nonzero 3c. the line $x = 2$ 3d. the line $y = x$ 4a. 5 4b. 1 4c. $\sqrt{13}$
4d. $\sqrt{13}$ 5. slope 2/3, slope $-6/4$

page 2

1a. $10 + 4i$ 1b. $-2 + 5i$ 1c. $-4 + 10i$ 2. sides $\sqrt{10}$, $\sqrt{8}$, $\sqrt{2}$ and $\sqrt{50}$, $\sqrt{40}$, $\sqrt{10}$; ratio of similarity $\sqrt{5}$; both triangles right 3a. $\text{cis}(53.1)$ 3b. $\text{cis}(120)$ 3c. $\text{cis}(73.7)$
4. $0.618 + 1.902i$; $5\text{cis}(53.1)$ 5. $5\text{cis}(53.1)$; $\sqrt{2}\text{cis}(45)$; $\sqrt{50}\text{cis}(98.1)$ 6. 56; $12\text{cis}(56)$
7. i ; -1 ; $-i$; 1 8. $(ac - bd) + (ad + bc)i$; addition formulas for cosine and sine 9. 13; $a^2 + b^2$ 10a. reflection across x -axis 10b. nonnegative real result; square of magnitude
10c. real result 10d. $\frac{(a + bi)(c + di)}{a + bi \cdot c + di} = \frac{(ac - bd) + (ad + bc)i}{(ac - bd) + (ad + bc)i} = (ac - bd) - (ad + bc)i = (a - bi)(c - di) = a + bi \cdot c + di$

page 3

1. -1 2. $\frac{1}{2}(1 - i)$ 3. $7.6 - 8.2i$ 4. 16 6a. -2^{18} 6b. $\text{cis}(\alpha)/\text{cis}(\beta) = \text{cis}(\alpha - \beta)$ 7. common-base exponents 8. $-(2 + i)$ 9. there are two square roots to name 10. $\frac{1}{2}(-i \pm 3)$ 11a. $\frac{1}{2}(-i \pm 3)$ 11b. 2 and $\frac{1}{2}(-i \pm 3)$ 11c. $0.6 - 0.8i$
12. $1 - i$, $-1 + i$, $-1 - i$ 13. 0 14. 33 15. $1 + i$ 16. $0.8 + 0.6i$

page 4

1a. $|z_{40}| = 1.012574i$ 1b. 57.2838° 1c. 252 2. $a = 12$, $b = -16$ 3. $\sqrt{wz} \neq \sqrt{w}\sqrt{z}$ 4. 233 5. $\pm(4 + i)$ 6a. 0 6b. 0 7. $(a^2 + b^2)(c^2 + d^2)$ 8. $133^2 + 41^2 = 19370 = 139^2 + 7^2$ 9. if $m = a^2 + b^2$ and $n = c^2 + d^2$, then $mn = (ac - bd)^2 + (ad + bc)^2$ 11. $25(1 + i)$ and $(2a^2 + 2a + 1)(1 + i)$ 12. 45

page 5

1. 1 , $\frac{1}{2}(-1 \pm i\sqrt{3})$; $\text{cis}(0)$, $\text{cis}(120)$, $\text{cis}(240)$ 2. $c_5(z) = z^4 + z^3 + z^2 + z + 1$ 3. $p(z) = (z + 1)(z^2 + 1)(z^2 + z + 1)(z^2 - z + 1)(z^4 - z^2 + 1)$ 5. 572 6. 15 7. 198
8. 315 9. conjugate both sides of the equation to find that z is a solution iff \bar{z} is a solution 10. 51 11. 259 12. 77 13. 177

page 6

1. 275 2. $(1 + b)/a$; period 5 3. x_1 ; period 9 [AMM 1985] 4. $f_1(2008) = 2007/2008$; period 3 5. 5 6. 15 7. (C) 8. (B) 9. (B) 10. (D)
11. 2

page 7

1a. 6 1b. 5 1c. denominators do not increase 2. 32; $\text{cis}(90)$, $\text{cis}(162)$, $\text{cis}(234)$, $\text{cis}(306)$ 3. 3 4. 4 5. 5 6. 6 7. 7 n. n 8. $z^4 - z^2 + 1$ 10. k^2
11. 1