

Primes and divisors

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IM Primes

- 1.1.1. The integer $p > 1$ is called a *prime* (or a *prime number*) if there is no integer d with $d > 1$ and $d \neq p$ such that $d \mid p$.
The first few primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29. Determine the number of primes less than 100.
- 1.1.2. Find all positive integers n for which $3n - 4$, $4n - 5$, and $5n - 3$ are all prime numbers.
- 1.1.3. If p and q are primes and $x^2 - px + q = 0$ has distinct positive integral roots, find p and q .
- 1.1.4. An integer $n > 1$ that is not a prime is called *composite*.
What is the smallest composite number with no prime factor less than 10?
- 1.1.5. Find 20 consecutive composite numbers.
- 1.1.6. Several set of prime numbers, such as $\{7, 83, 421, 659\}$, use each of the nine nonzero digits exactly once. What is the smallest possible sum such a set of primes could have?
- 1.1.7. (Continuation) How many such sets of primes have their sums equal to this minimum possible sum?
- 1.1.8. Determine all integers n such that number $n^4 + 4$ is prime.
- 1.1.9. Find all solutions (n, p) with n a positive integer and p a prime satisfying the equation: $n^4 - 16n^2 = p^2 + 2p - 63$.
- 1.1.10. Prove that there are infinitely many primes.

Divisors

- 1.1.11. Five points on a circle are numbered 1, 2, 3, 4, and 5 in clockwise order. A bug jumps in a clockwise direction from one point to another around the circle; if it is on an odd-numbered point, it moves one point, and if it is on an even-numbered point, it moves two points. If the bug begins on point 5, after 1995 jumps what point will it be on?
- 1.1.12. Numbers a, b, c, d , and e are chosen randomly (with repetitions allowed) from the set $S = \{1, 2, 3, 4, 5\}$. Given that the probability that $abcd + e$ is odd is $\frac{m}{n}$, where m and n are relative prime positive integers, find $m + n$.
- 1.1.13. Zach has chosen five numbers from the set $\{1, 2, 3, 4, 5, 6, 7\}$. If he told Claudia what the product of the chosen numbers was, that would not be enough information for Claudia to figure out whether the sum of the chosen numbers was even or odd. What is the product of the chosen numbers?
- 1.1.14. In how many ways may 1, 2, 3, 4, 5, and 6 be ordered so that no two consecutive terms have a sum that divisible by 2 or 3?
- 1.1.15. Compute sum of the greatest odd divisor of each of the numbers 2006, 2007, \dots , 4012.
- 1.1.16. Twenty bored students take turns walking down a hall that contains a row of closed lockers, numbered 1 to 20. The first student opens all the lockers; the second student closes all the lockers numbered 2, 4, 6, 8, 10, 12, 14, 16, 18, 20; the third student operates on the lockers numbered 3, 6, 9, 12, 15, 18: if a locker was closed, he opens it, and if a locker was open, he closes it; and so

on. For the i^{th} student, he works on the lockers numbered by multiples of i : if a locker was closed, he opens it, and if a locker was open, he closes it. What is the number of the lockers that remain open after all the students finish their walks?

1.1.17. Let $(a_1, a_2, \dots, a_{99})$ be a permutation of $(1, 2, \dots, 99)$. Show that the product

$$(a_1 - 1)(a_2 - 2) \cdots (a_{99} - 99)$$

is even.

1.1.18. If n is a composite integer, explain that it has a prime divisor p not exceeding \sqrt{n} . (This idea belongs to the ancient Greek mathematician Eratosthenes, 250 BCE.)

1.1.19. Find the number of zeros at the end of the decimal representation of $100!$.

1.1.20. Call a number *prime looking* if it is composite but not divisible by 2, 3, or 5. The three smallest prime-looking numbers are 49, 77, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?

Exercises

1.1.21. Integers a, b, c , and d , not necessarily distinct, are chosen independently and at random from 0 to 2007, inclusive. What is the probability that $ad - bc$ is even?

1.1.22. Find all ordered pairs (m, n) of positive integers such that $m^2 - n^2 = 150$.

1.1.23. If a, b and c are three (not necessarily distinct) numbers chosen randomly and with replacement from the set $\{1, 2, 3, 4, 5\}$, what is the probability that $ab + c$ is even?

1.1.24. Determine the sum of all integers n such that $n^2 - 14n + 24$ is prime.

1.1.25. Determine the number quadratic polynomials $p(x)$ with integer coefficients and with two distinct integer roots such that $p(8) = 6$.

1.1.26. Six distinct positive integers are randomly chosen between 1 and 2006, inclusive. What is the probability that some pair of these integers has a difference that is a multiple of 5?

1.1.27. Two farmers agree that pigs are worth \$300 and the goats are worth \$210. When one farmer owes the other money, he pays the debt in pigs or goats, with “change” received in the form of goats or pigs as necessary. (For example, a \$390 debt could be paid with pigs, with one goat received in change.) What is the amount of the smallest positive debt that can be resolved in this way?

1.1.28. Compute the number of sets of five consecutive integers in $\{1, 2, \dots, 50\}$ such that the product of the elements of each set of five integers is divisible by both 12 and 21.

1.1.29. Given a sequence of six strictly increasing positive integers, such that each number (besides the first) is a multiple of the one before it and the sum of all six numbers is 79, what is the largest number in the sequence?

1.1.30. On a 75×75 board, rows and columns are numbered 1 through 75. Alex wants to put a pawn in exactly the squares with an even coordinate and the other a multiple of 3. How many pawns are needed?